1. Introduction

One of the important strategies of improving the operational implementation of 4DVar is to develop an ensemble-based 4DVar (ED4Vf). This can be achieved by transforming the 4DVar ensemble dimension into a projection onto the subspace spanned by the observed covariances. The dimension-reduced projection 4DVar (DRP-4DVar) proposed by Wang et al. (2010) is one representative approach of the ED4Vf family. The DRP-4DVar solves the 4DVar problem in the subspace defined by an ensemble. This approach conserves the advantages of the 4DVar while assisting in the construction of the adjoint model. Thus, DRP-4DVar is a much more computationally economic method to implement 4DVar than the standard 4DVar. DRP-4DVar uses real-time ensemble to estimate the background error covariance outside the assimilation window, which ensures the global flow dependency. Plenty of studies on DRP-4DVar have shown good performance both with simple models like Lorenz96 and realistic models like ENS. However, DRP-4DVar has a potential problem. The DRP-4DVar replaces the target linear model 4DVar with a linear statistic relation to forecast perturbations 4DVar. Thus, DRP-4DVar results in the model and cost function is simplified as an explicit quadratic polynomial. As a result, the performance of DRP-4DVar is governed by the linear approximation. This work focuses on this potential problem. We verify the problem by Lorenz96 model and present an iterative nonlinear correction process to alleviate the problem.

2. Method

2.1. DRP-4DVar

For 4DVar, its cost function can be written as follows:

\[ \min \{ J(\mathbf{X}) \} \]

where \( \mathbf{X} \) is the total number of observation times within the assimilation window, \( \{ \mathbf{Y}_t \} \) is the sequence of observations.

In DRP-4DVar, we have two possible types of initial condition perturbations: \( \mathbf{P} \) and \( \mathbf{u} \). If we assume the analysis increment can be expressed as a linear combination of the ensemble members (i.e., \( \mathbf{u} = \mathbf{x} \mathbf{A} \mathbf{P} \)), we can write the cost function transformation as a function of \( \mathbf{a} \):

\[ J_{\text{DRP-4DVar}}(\mathbf{a}) = \sum_{t=1}^{T} \left( \mathbf{Y}_t - \mathbf{X}_t \right)^T \mathbf{K}^{-1} \left( \mathbf{Y}_t - \mathbf{X}_t \right) \]

where \( \mathbf{K}^{-1} \) is the observational error covariance and \( \mathbf{X}_t \) is the ensemble transformation of the target linear model.

A linear approximation could be the variety of from Taylor expansion: \( \mathbf{X}_t^1 \approx \mathbf{X}_t - \mathbf{K}^{-1} \mathbf{Y}_t \).

As in Fig. 2, we can see the two points of NC-DRP-4DVar: the original DRP-4DVar is a special case of NC-DRP-4DVar with one outer loop. The strategy to update ensemble \( \mathbf{P} \) is understood. Two strategies are tested here:

1. \( \mathbf{P} \) is updated in each outer loop; no actual updating is performed, so it’s economical but empirical
2. \( \mathbf{P} \) is updated in one outer loop; re-initialized, it’s theoretically correct but very time-consuming!

3. Experiments basic settings

3.1. Observation system simulation experiment (OSSE)

- Lorenz model

\[ \dot{X} = \sigma (Y-X), \quad \dot{Y} = RX - Y - XZ, \quad \dot{Z} = XY - BZ, \quad \sigma = 10, R = 28, B = 8/3 \]

- Forecast period: \( T = 100 \) days

- Assimilation period: 4 days

- Observations: spatial distribution = \( \{1,2, \ldots, 30\} \), \( F \) = 4 days

- Observations error: standard deviation: \( \sigma (i) = 0.5 \)\% (bias-free)

- Assimilation interval: 4 days

- Observations: perturbed the truth with uncorrelated Gaussian noise (variance = 0.16), available all at grid 2 (begin of the window)

- Ensemble number: \( n = 100 \) (ensemble number is sufficient or to estimate \( \mathbf{K} \) correctly)

3.2. Experiments

3.2.1. Assimilation Schemes

- 4DVar (incremental approach) \( \approx \) adjoint model, inner loop: conjugate gradient method, outer loop: \( \mathbf{P} \) reduces more than 90\%, max iterations \( = 12 \), outer loop iterations \( = 5 \)

- DRP-4DVar: solution \( P \) of (a); \( \mathbf{P} \) of DRP-4DVar related schemes is randomly generated with Gaussian distribution (zero mean variance among all windows). \( \mathbf{P} \) of DRP-4DVar related schemes is also inflated by \( \mathbf{I} \), i.e., from ‘perfect’ inflation: \( \kappa \mathbf{P} \) (b), \( \kappa \mathbf{P} \) is a diagonal matrix that approximates the variance of \( \mathbf{K} \). To further reduce computational cost, an EOF decomposition is performed on \( \mathbf{P} \), before assimilation and the number of EOF modes selected is set to 40.

3.3. Experiments

4. Experiments – CTRLB

Design: a set of 30 windows assimilation experimental background in from a control run (starts with the initial background and integrates freely for 10 windows), so as assimilations; two strategies to update the ensemble are introduced: a) ‘Re-Integrating’ strategy, b) ‘No Updating’ strategy. These two strategies to update the ensemble are introduced: a) ‘Re-Integrating’ strategy, b) ‘No Updating’ strategy. These two strategies are tested in the experiment, where ‘Re-Integrating’ only updates the ensemble once and re-assimilates in the rest outer loops while use ‘No Updating’ in the rest outer loops, namely ‘Re-Integrating’ strategy, its RMSE is almost the same with that of 4DVar, which proves our theoretical analysis. Therefore, the nonlinear correction process with the DRP-4DVar scheme can improve its performance.

4.2. Experiments – CYCLEDA

Cycling assimilation

5. Experiments – CYCLEDA

Cycled CYCLEDA uses various updated background ensemble or the ensemble of the DRP-4DVar related schemes, i.e. the global flow-dependent covariance, (GR) of the cycle and the observation error covariance. In the cycle, the ensemble is updated in a circular manner, so that the cycle of ensemble-based covariances of DRP-4DVar, multiplied with a nudging factor. As the background of good quality is of great importance for the experiment, the nonlinear correction is employed, and only the ‘No-Integrating’ strategy is applied.

5.1. Experiments

5.1.1. Experiments

In this study, we prove that the DRP-4DVar is limited by its linear approximation and therefore we extend it with a nonlinear correction process, forming the NC-DRP-4DVar. In the im-plementation of the NCDRP-4DVar, we use the Re-Integrating strategy, and the linear approximation is replaced by the nonlinear correction process. The nonlinear correction process is performed by the analysis increment with a nonlinear correction process, forming the NCDRP-4DVar, two strategies to update the ensemble are introduced: a) ‘No Updating’ strategy, b) ‘Re-Integrating’ strategy. The latter proves to be more significant improvements due to the correction of nonlinear effect, especially for the ‘Re-Integrating’ strategy, as in Fig. 3, NCDRP-4B gets a similar performance with NCDRP-4B while it only updates the ensemble once and re- assimilates in the rest outer loops of the cycle of the computation of NCDRP-4B.

6. Summary

In this paper, we prove that the DRP-4DVar is limited by its linear approximation and therefore we extend it with a nonlinear correction process, forming the NC-DRP-4DVar. In the implementation of the NCDRP-4DVar, we use the Re-Integrating strategy, and the linear approximation is replaced by the nonlinear correction process. The nonlinear correction process is performed by the analysis increment with a nonlinear correction process, forming the NCDRP-4B gets a similar performance with NCDRP-4B while it only updates the ensemble once and re-assimilates in the rest outer loops of the cycle of the computation of NCDRP-4B.

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